Alias-Free, Real-Coefficient m-Band QMF Banks for Arbitrary m

P. P. VAIDYANATHAN, MEMBER, IEEE, AND KUMAR SWAMINATHAN, MEMBER, IEEE

Abstract —Based on a generalized framework for alias free QMF banks, a theory is developed for the design of uniform QMF banks with real-coefficient analysis filters, such that aliasing can be *completely* canceled by appropriate choice of real-coefficient synthesis filters. These results are then applied for the derivation of closed-form expressions for the synthesis filters (both FIR and IIR), that ensure cancelation of aliasing for a given set of analysis filters. The results do not involve the inversion of the alias-component (AC) matrix.

I. Introduction

IN THIS PAPER we consider m-channel, maximally decimated parallel QMF banks shown in Fig. 1, where $H_k(z)$ and $F_k(z)$ are the analysis and synthesis filters. The signals which are outputs of the analysis filters are decimated by a factor of m and transmitted after possible coding. At the synthesis end, the sampling rate is increased by m, and the signals are filtered through $F_k(z)$ and combined to produce $\hat{x}(n)$. Such analysis/synthesis systems find applications in subband coding and in other areas [1]–[4].

A common requirement in most applications is that the reconstructed signal $\hat{x}(n)$ should be "as close to $x(n)$ as possible" in a certain well defined sense. The most general relation between $x(n)$ and $\hat{x}(n)$ is conveniently expressed in the transform domain [3], [14], [16] as

$$\hat{X}(z) = \sum_{l=0}^{m-1} X(zW_m^{-l}) \sum_{k=0}^{m-1} H_k(zW_m^{-l}) F_k(z) \Big/ m \quad (1)$$

where W_m stands for $e^{-j2\pi/m}$. The terms in (1) corresponding to $l \neq 0$ represent aliasing (caused by decimation of the filtered signals in the analysis bank). If the analysis and synthesis filters are chosen to cancel aliasing, the result is

$$\hat{X}(z) = T(z)X(z), \quad T(z) = \sum_{k=0}^{m-1} H_k(z)F_k(z)\Big/m. \quad (2)$$

Thus once aliasing is canceled, the linear system of Fig. 1 becomes shift invariant as well, with transfer function $T(z)$ which is also called the "overall transfer function" or the "distortion transfer function." If an alias free system satisfies $T(z) = cz^{-n_0}$ where c is some constant and n_0 is a positive integer, the QMF bank is said to be a perfect reconstruction system.

For the two channel case, exact methods for alias cancelation [1]–[4] and for perfect reconstruction [5] are known. For the m-channel case with arbitrary m, very useful methods for approximate cancelation of aliasing have been reported [8], [9]. Methods for *exact* cancelation of aliasing in the case of uniform DFT analysis banks [3] have been reported in [13] recently. Such exact cancellation requires the synthesis filters to be of typically much higher order than the analysis filters. Next, techniques for perfect reconstruction for arbitrary m have also been reported earlier [5], [14], [10], [17]. Some of these are based on the inversion of the alias component (AC) matrix [5], [14] while some others [17] are based on unitary AC matrices and lead to equal-length FIR analysis and synthesis filters.

In this paper we consider QMF banks as in Fig. 1 where the analysis filters $H_k(z)$ have real coefficients and hence $|H_k(e^{j\omega})|$ has two sidebands (i.e., is symmetric with respect to $\omega = \pi$. Accordingly, the superscript DSB is used (for instance $H_0^{\mathrm{DSB}}(z)$) while referring to the analysis transfer functions. Such transfer functions can be obtained by combining appropriate pairs of transfer functions in a complex-coefficient QMF bank (for example a uniform generalized-DFT (abbreviated GDFT) bank [3]). This results in a set of analysis filters that are cosine-modulated versions of a prototype impulse response. Fig. 2(a) shows a typical prototype response, and Fig. 2(b) shows the response of the analysis filters for the case of $m = 4$. Notice that $|H_0^{\mathrm{DSB}}(e^{j\omega})|$ is not centered around zero frequency.

Now, for a complex coefficient (uniform DFT) QMF analysis bank, complete cancellation of aliasing can be achieved by choice of synthesis filters as described in [13]. If the analysis filters in such a bank are combined to obtain real-coefficient filters, how should the filters in the synthesis bank be combined so as to retain the alias-free property? The answer to this, which is not obvious, is the main topic of this paper. The problem addressed in this paper differs from the perfect reconstruction problem addressed in [17] for the following reasons: first, in this paper we are primarily interested in canceling aliasing distortion alone. The resulting distortion function $T(z)$ can be forced to be an all-pass function (if amplitude distortion cannot be tolerated) or to be a linear phase FIR function (if phase distortion cannot be tolerated). The distortion that remains after this should either be minimized by optimization or *equalized*. On the other hand, the problem addressed in

Manuscript received May 5, 1986; revised February 25, 1987. This work was supported in part by the National Science Foundation under Grant DCI 8552579, and in part by Caltech's Programs in Advanced Technology Grant sponsored by Aerojet General, General Motors, GTE, and TRW.

P. P. Vaidyanathan is with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125.

K. Swaminathan was with California Institute of Technology, Pasadena, CA. He is now with AT&T Bell Laboratories, Whippany, NJ 07981.

IEEE Log Number 8717303.

0098-4094/87/1200-1485$01.00 ©1987 IEEE

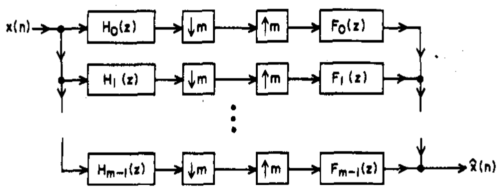


Fig. 1. The m-band maximally decimated QMF bank.

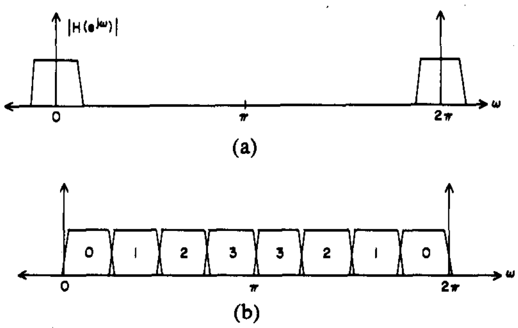


Fig. 2. (a) Magnitude response of the baseband prototype filter; (b) Magnitude response for the modulated filters $H_k^{\mathrm{DSB}}(z)$, with $m=4$.

[17] is the perfect reconstruction problem, i.e., not only is aliasing canceled, but in addition $T(z)$ is a pure delay. The designs in [17] are accordingly restricted to the case where the AC matrix is paraunitary. The analysis filters in [17] are not simple cosine-modulated versions of a prototype, but are designed based on a special lattice optimization procedure. In the present paper, we are looking only for alias cancelation, which permits more flexible designs of the analysis bank; our main aim in this paper is to derive closed-form expressions for synthesis filters that would *completely* cancel aliasing for an *arbitrary* analysis prototype (the analysis filters being cosine modulated versions of the prototype).

The synthesis filters we derive in this paper have much higher order than the corresponding analysis filters for large m, just as in [13] (unlike the perfect reconstruction systems in [17]). As an illustration, the design example in Section VII of this paper (three-band QMF bank) has analysis filters $H_k(z)$ of order 55, whereas $F_k(z)$ have order 267. The cost of the analysis bank is approximately $56c$ (where c is constant) and that of the synthesis bank is $268c$. In general, since $F_k(z)$ have higher order than $H_k(z)$, the cost of the synthesis bank is higher than that of the analysis bank. However, it is possible to implement these in an efficient polyphase form [3], because of the cosine-modulated nature of the filters. Moreover, one can build tree structures with building blocks having $m=2,3$ which is more general than binary trees.

Now, it can be shown [18] that, if a QMF bank is free from aliasing and has distortion function $T(z)$ as given in (2), it remains alias-free (with same $T(z)$) if each $H_k(z)$ is interchanged with the corresponding $F_k(z)$. (This is called the *swapping property*). If $H_k(z)$ and $F_k(z)$ are swapped, we have a high-cost analysis bank and a low-cost synthesis bank. Such manipulations are useful in *mobile communications systems* where there is one stationary transmitter, and several mobile receiving stations. For example, consider the three-band system mentioned above, with $H_k(z)$ and $F_k(z)$ swapped. If there is one fixed station and fifty mobile stations, the total cost is $268c+2800c$ which is dominated by the cost of the synthesis filters. In other words, even though *exact cancelation* of aliasing appears to be expensive in applications with one transmitter and one receiver, this is not so in mobile applications, if the mobile stations are made the inexpensive ones. The extra overhead cost, caused by the attempt to cancel aliasing perfectly, is a small fraction of the total cost of the mobile network.

Section II presents a general m-band alias-free system, which will be used to build up several useful alias-free QMF banks based on uniform DFT and GDFT blocks. In Section III we obtain the analysis filters $H_k^{\mathrm{DSB}}(z)$ by using the alias free system of Section II. This involves the addition of pairs of filters (in a uniform GDFT bank) with complex conjugate coefficients. We then outline the overall logic for deriving expressions for synthesis filters $F_k^{\mathrm{DSB}}(z)$ which cancel aliasing. The actual derivations are included in Section IV. Section V simplifies the solution of Section IV, whereas Section VI further specializes the results to specific forms suitable for FIR and stable IIR synthesis banks. Finally, Section VII includes a design example.

The derivations here are admittedly more complicated than the counterpart in [13], due to the fact that we do not have a *circulant* AC matrix here (which leads to nice simplifications in [13]). The reason for splitting up our main results here into Sections III–V is our hope that, this will make the paper relatively easy to read. Certain important properties of filter banks, that we use in the paper are listed in the Appendix without proof, in an attempt to offer a smoother reading. Further details and proofs can be found in [16].

Notations: Boldfaced letters denote matrices and vectors; $\boldsymbol{I}_k$ is the $k\times k$ identity matrix; $\boldsymbol{0}$ is the null matrix of appropriate dimensions. The matrix $\boldsymbol{J}_k$ is obtained from $\boldsymbol{I}_k$ by reversing the columns; for example,

$$\boldsymbol{J}_3=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Superscript T stands for matrix-transposition. The tilde accent on a scalar or matrix function of z (such as $\tilde{\boldsymbol{H}}(z)$) indicates transposition, followed by conjugation of coefficients (if complex), followed by replacement of z with z^{-1}. On the unit circle, this is simply transposed-conjugation. The notation W_r stands for $e^{-j2\pi/r}$ and $\boldsymbol{W}_r$ is the $r\times r$ DFT matrix, i.e., $\boldsymbol{W}_r=[W_r^{kl}]$. The generalized DFT (GDFT) of an r-point sequence y_k is defined to be [3], $Y_l=\sum_{k=0}^{r-1}y_kW_r^{(l+l_0)(k+k_0)}$, for $0\leqslant l\leqslant r-1$. The inverse GDFT is defined by $y_k=\sum_{l=0}^{r-1}Y_lW_r^{-(l+l_0)(k+k_0)}/r$, $0\leqslant k\leqslant r-1$.

II. Alias-Free, Parallel QMF Banks

Many useful alias-free, maximally decimated QMF banks can be derived by referring to the model of Fig. 3. In this arrangement, there are r transfer functions $S_k(z)$, $0\leqslant k\leqslant$

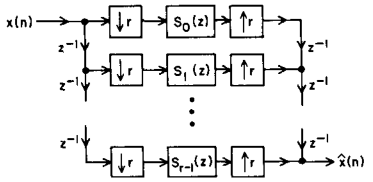


Fig. 3. An r-band system which is alias-free if and only if $S_k(z) = S(z)$ for all k.

$r-1$ sandwiched between the r-fold decimators and interpolators. The decimators are preceded by a chain of $r-1$ delays, and the interpolators are followed up with a delay chain. It can be verified that the output signal of such a system is given by

$$\hat{X}(z) = z^{-(r-1)} \sum_{l=0}^{r-1} X(zW_r^l) \sum_{k=0}^{r-1} W_r^{-lk} S_k(z^r)/r \quad (3)$$

where the terms with $l \neq 0$ are the aliasing terms. These terms are absent (i.e., $\hat{x}(n)$ is free from aliasing), if and *only if* the functions $S_k(z)$ are independent of k, i.e., $S_k(z) = S(z)$, $0 \leq k \leq r-1$ (see Appendix A). Under this condition we have $\hat{X}(z) = T(z)X(z)$ where

$$T(z) = z^{-(r-1)} S(z^r) \quad (4)$$

is the distortion transfer function of the system. Now consider r arbitrary factorizations of $S(z)$ in the form $S(z) = G_l(z)R_l(z)$, $0 \leqslant l \leqslant r-1$. The alias free system of Fig. 3 can then be redrawn, as in Fig. 4(a), by using standard multirate identities [3]. If we now insert an arbitrary nonsingular matrix $\boldsymbol{T}$ and its inverse into the system as in Fig. 4(b), the output $\hat{x}(n)$ is unaffected. In conclusion, Fig. 4(b) represents a general alias-free system if and only if $G_l(z)R_l(z)$ is independent of l. Under this condition the overall distortion transfer function is given by (4), with $S(z) = G_l(z)R_l(z)$.

If we take $\boldsymbol{T} = \boldsymbol{W}_r/r$, then Fig. 4(b) represents a uniform DFT QMF bank [3], [13]. Similarly, we can construct a uniform GDFT QMF bank by taking $\boldsymbol{T} = \boldsymbol{W}_r^{\mathrm{GDFT}}/r$. For such a system, it is easily shown that the analysis filters $H_k^{\mathrm{GDFT}}(z)$ are related by

$$H_k^{\mathrm{GDFT}}(z) = W_r^{-l_0 k} H_0^{\mathrm{GDFT}}(zW_r^k) \quad (5)$$

with

$$H_0^{\mathrm{GDFT}}(z) = W_r^{-l_0 k_0} \sum_{l=0}^{r-1} G_l^{\mathrm{GDFT}}(z^r)(zW_r^{k_0})^{-l}. \quad (6)$$

Similarly the synthesis filters are given by

$$F_k^{\mathrm{GDFT}}(z) = W_r^{k(l_0+r-1)} F_0^{\mathrm{GDFT}}(zW_r^k)$$

where

$$F_0^{\mathrm{GDFT}}(z) = W_r^{k_0(l_0+r-1)} \sum_{l=0}^{r-1} R_l^{\mathrm{GDFT}}(z^r) \cdot (zW_r^{k_0})^{-(r-1-l)}/r. \quad (7)$$

The superscripts on $G_l(z)$ and $R_l(z)$ in (6), (7) are meant to be a reminder that the GDFT bank is currently under discussion. The analysis filter magnitude-responses are, therefore, uniformly shifted versions of a prototype, and so are the synthesis filter magnitude-responses. These filters can be expressed directly in terms of the components $G_l^{\mathrm{GDFT}}(z)$ and $R_l^{\mathrm{GDFT}}(z)$ as

$$H_k^{\mathrm{GDFT}}(z) = W_r^{-l_0 k_0} W_r^{-l_0 k} \sum_{l=0}^{r-1} G_l^{\mathrm{GDFT}}(z^r)(zW_r^{k_0}W_r^k)^{-l} \quad (8)$$

and

$$F_k^{\mathrm{GDFT}}(z) = W_r^{l_0(k_0+k)-k} \cdot \left[\sum_{l=0}^{r-1} R_l^{\mathrm{GDFT}}(z^r)(zW_r^{k+k_0})^{-(r-1-l)}\right] W_r^{k_0(r-1)}/r. \quad (9)$$

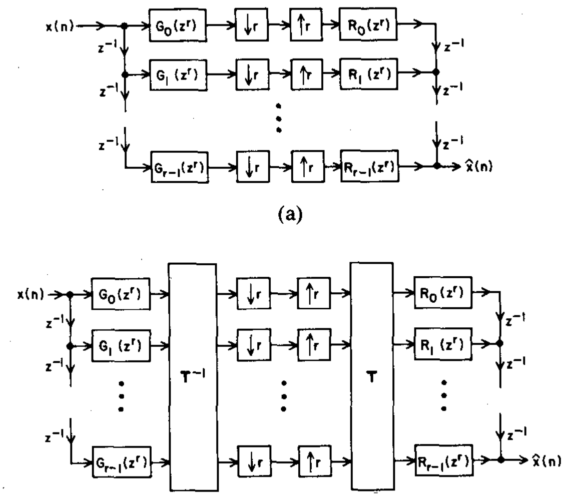
If we set $k_0 = l_0 = 0$ in the above expressions, we obtain the corresponding relations for a standard uniform DFT bank. For example, we have

$$H_k^{\mathrm{DFT}}(z) = \sum_{l=0}^{r-1} G_l^{\mathrm{DFT}}(z^r)(zW_r^k)^{-l} \quad (10)$$

and

$$F_k^{\mathrm{DFT}}(z) = W_r^{-k} \sum_{l=0}^{r-1} R_l^{\mathrm{DFT}}(z^r)(zW_r^k)^{-(r-1-l)}/r. \quad (11)$$

Typical responses for a uniform DFT bank are shown in Fig. 5(a). The main motivation for using GDFT rather than DFT banks is to provide a frequency shift for the responses so that $|H_k^{\mathrm{GDFT}}(e^{j\omega})|$ is an image of $|H_{r-1-k}^{\mathrm{GDFT}}(e^{j\omega})$ for each k. In this way, $H_k^{\mathrm{GDFT}}(z)$ and $H_{r-1-k}^{\mathrm{GDFT}}(z)$ can be combined to produce $H_k^{\mathrm{DSB}}(z)$ having real coefficients. Given a uniform DFT filter bank, we can create a GDFT bank simply by replacing each z with $zW_r^{k_0}$, with k_0 chosen to give the desired frequency shift. The usefulness of the additional parameter l_0 will be clarified in Section III.



(a)

(b)

Fig. 4. (a) A "factored version" of Fig. 3. (b) Inserting nonsingular $\boldsymbol{T}$ into Fig. 4(a).

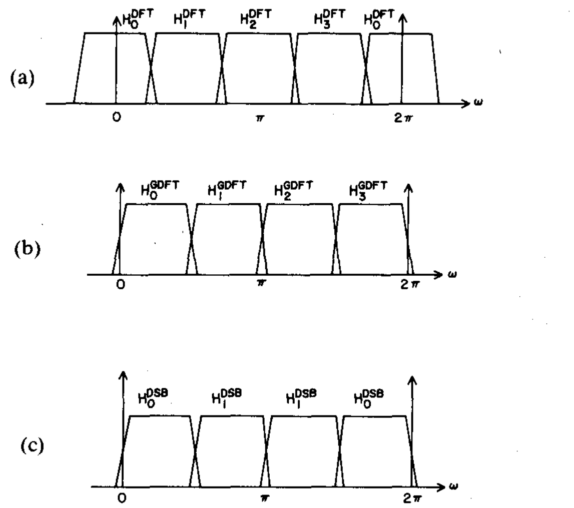


Fig. 5. Typical magnitude responses for analysis filters, with $m=2$, $r=2m=4$. (a) The uniform DFT bank. (b) The uniform GDFT bank. (c) The uniform DSB bank.

In any case, given an alias free uniform DFT QMF bank, we can obtain an alias free uniform GDFT QMF bank by replacing z with $zW_r^{k_0}$, multiplying the kth analysis filter with $W_r^{-(k+k_0)l_0}$ and multiplying the kth synthesis filter with $W_r^{(k+k_0)l_0+k_0(r-1)}$. The analysis filters of the GDFT bank so obtained are related to the corresponding DFT bank prototype filter $H_0^{\mathrm{DFT}}(z)$ by

$$H_k^{\mathrm{GDFT}}(z)=W_r^{-l_0(k+k_0)}H_0^{\mathrm{DFT}}\left(zW_r^{k+k_0}\right). \tag{12}$$

Fig. 5(b) shows the shifting operation caused by (12). Similarly,

$$F_k^{\mathrm{GDFT}}(z)=W_r^{k_0(r-1)+l_0(k+k_0)-k}F_0^{\mathrm{DFT}}\left(zW_r^{k+k_0}\right). \tag{13}$$

Notice that $G_l^{\mathrm{DFT}}(z)$ are precisely the polyphase components [3] of $H_0^{\mathrm{DFT}}(z)$ whereas $G_l^{\mathrm{GDFT}}(z)$ are related to $H_0^{\mathrm{GDFT}}(z)$ in a more complicated way. The impulse response coefficients of $G_l^{\mathrm{DFT}}(z)$ can be obtained by decimating appropriately shifted versions of the impulse response coefficients of $H_0^{\mathrm{DFT}}(z)$. It is worth noting that the alias-free DFT and GDFT banks satisfy the relations

$$\begin{aligned} G_k^{\mathrm{GDFT}}(z^r)&=G_k^{\mathrm{DFT}}\left(z^rW_r^{k_0r}\right)\\ R_k^{\mathrm{GDFT}}(z^r)&=R_k^{\mathrm{DFT}}\left(z^rW_r^{k_0r}\right). \end{aligned} \tag{14}$$

The distortion transfer functions $T(z)$ of the alias free DFT and GDFT banks are related by

$$T^{\mathrm{GDFT}}(z)=W_r^{k_0(r-1)}T^{\mathrm{DFT}}\left(zW_r^{k_0}\right). \tag{15}$$

III. m-Band Cosine Modulated QMF Banks from $2m$-Band GDFT Banks

The analysis filters $H_k^{\mathrm{GDFT}}(z)$ in the uniform GDFT bank are derived from a prototype $H_0^{\mathrm{DFT}}(z)$, whose impulse response $h_0^{\mathrm{DFT}}(n)$ is real. From (12) we have

$$h_k^{\mathrm{GDFT}}(n)=h_0^{\mathrm{DFT}}(n)W_r^{-(l_0+n)(k+k_0)}. \tag{16}$$

Our aim is to add the impulse responses of $H_k^{\mathrm{GDFT}}(z)$ and $H_{r-1-k}^{\mathrm{GDFT}}(z)$ to obtain the *real valued* impulse response $h_k^{\mathrm{DSB}}(n)$ of $H_k^{\mathrm{DSB}}(z)$. In this way, the r bands of the GDFT bank are combined into $r/2=m$ bands (see Fig. 5(c); for the rest of the paper we shall assume that r is an even number given by $r=2m$). Now the sum of $h_k^{\mathrm{GDFT}}(n)$ (given by (16)) and $h_{2m-1-k}^{\mathrm{GDFT}}(n)$ (given by a similar expression) will be real, if the exponents of W_r in $h_k^{\mathrm{GDFT}}(n)$ and $h_{2m-1-k}^{\mathrm{GDFT}}(n)$ add up to a multiple of $2m$, i.e., if

$$(l_0+n)(2k_0+2m-1)=\text{integer multiple of } 2m. \tag{17}$$

It is easily verified that (17) holds if k_0 is taken as $0.5 \bmod m$ for even $N-1$, and $(0.5+m) \bmod 2m$ for odd $N-1$. Thus the choice

$$k_0=(1/2)+m \tag{18}$$

works for all N. For the rest of the paper we shall restrict k_0 to be as in (18). With this choice of k_0 let us define

$$h_k^{\mathrm{DSB}}(n)=e^{j\theta_k}h_k^{\mathrm{GDFT}}(n)+e^{-j\theta_k}h_{2m-1-k}^{\mathrm{GDFT}}(n) \tag{19}$$

where θ_k is a real number, whose purpose will be explained later. From (16) and (19) we obtain

$$h_k^{\mathrm{DSB}}(n)=2h_0^{\mathrm{DFT}}(n)\cos\left\{\frac{2\pi}{2m}(l_0+n)(k+k_0)+\theta_k\right\}, \quad 0\leqslant k\leqslant m-1 \tag{20}$$

and (see Appendix C, Property 5)

$$H_k^{\mathrm{DSB}}(z)=2\sum_{l=0}^{2m-1}G_l(z^{2m})z^{-l}\cdot\cos\left\{\frac{2\pi}{2m}(l+l_0)(k+k_0)+\theta_k\right\}, \quad 0\leqslant k\leqslant m-1. \tag{21}$$

For simplicity, the superscript GDFT on $G_l(z)$ and $R_l(z)$ will often be deleted. Fig. 6 shows the structure of the resulting m-band analysis bank, where the $m\times 2m$ matrix $C=[C_{kl}]$ is defined by

$$C_{kl}=\frac{1}{m}\cos\left\{\frac{2\pi}{2m}(l+l_0)(k+k_0)+\theta_k\right\}, \quad 0\leqslant k\leqslant m-1;\quad 0\leqslant l\leqslant 2m-1. \tag{22}$$

For a $2m$-band maximally decimated GDFT QMF bank, if the transfer functions $R_l(z)$ are chosen such that $G_l(z)R_l(z)$ is independent of l, then the system is alias free, according to Section II. We have now created an m-band real-coefficient analysis bank by generating the analysis filters $H_k^{\mathrm{DSB}}(z)$. How should we choose the synthesis transfer functions $F_k^{\mathrm{DSB}}(z)$ of the DSB QMF bank, so as to cancel aliasing?

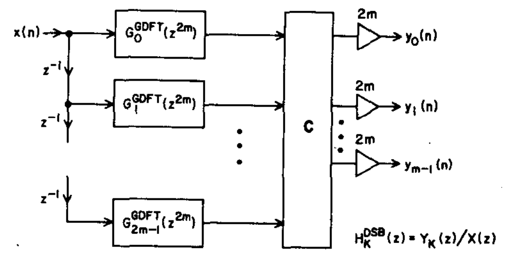


Fig. 6. Obtaining the cosine-modulated m-band DSB analysis bank.

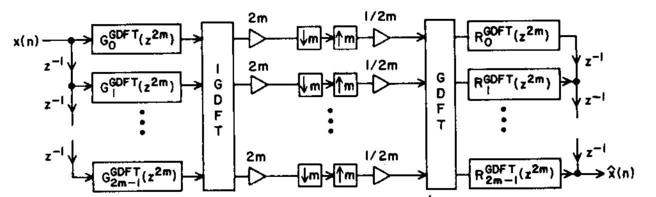


Fig. 7. The $2m$-band GDFT QMF bank with m-fold decimators and interpolators.

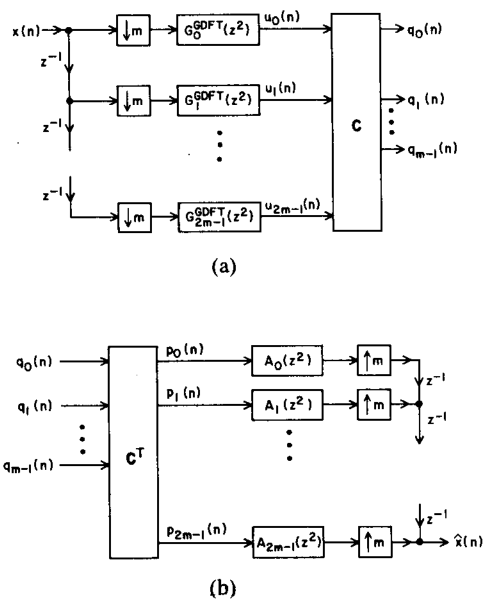


Fig. 8. (a) The cosine-modulated m-channel DSB analysis bank, redrawn for polyphase implementation. (b) The polyphase structure for the m-channel DSB synthesis bank.

First notice that, the decimation and interpolation factors are equal to $2m$ in the GDFT bank, but only m in the desired DSB bank. This, however, does not bother us because, if the decimators and interpolators in a $2m$ channel alias free QMF bank are replaced with m-fold counterparts, then it remains alias free, and the distortion transfer function gets multiplied by 2 (see Appendix B). In other words, the nonmaximally decimated GDFT QMF bank of Fig. 7 is alias free if

$$S(z) = R_l(z)G_l(z) \tag{23}$$

is independent of l for $0 \leqslant l \leqslant 2m-1$, and the distortion transfer function then is

$$T(z) = 2z^{-(2m-1)}S(z^{2m}). \tag{24}$$

We will, therefore, use Fig. 7 as a tool not only for generating the analysis bank for the m-band DSB QMF system, but also for obtaining the m synthesis filters $F_k^{\text{DSB}}(z)$ which make it alias free.

The m-fold decimators which follow the signals $y_k(n)$ in Fig. 6 can be moved to the left by use of standard multirate identities [3], resulting in the m-band DSB analysis bank structure of Fig. 8(a). Since we know how to design the $2m$-band alias-free GDFT QMF bank of Fig. 7, we know how to construct the alias-free signal $\hat{x}(n)$ of Fig. 7, starting from the signals $u_l(n)$, $0 \leqslant l \leqslant 2m-1$ in Fig. 8(a). However, in the DSB bank, the signals $q_l(n)$, $0 \leqslant l \leqslant m-1$ are being transmitted, hence we wish to construct the same $\hat{x}(n)$ as in Fig. 7, but starting from $q_l(n)$. Such reconstruction will show us how to design the synthesis bank corresponding to the m-band DSB analysis bank of Fig. 8(a). Defining the column vectors

$$\begin{aligned} \boldsymbol{q}(n) &= [q_0(n) \quad q_1(n) \cdots q_{m-1}(n)]^T \\ \boldsymbol{u}(n) &= [u_0(n) \quad u_1(n) \cdots u_{2m-1}(n)]^T \end{aligned} \tag{25}$$

we have $\boldsymbol{q}(n) = \boldsymbol{C}\boldsymbol{u}(n)$. Since $\boldsymbol{C}$ is $m \times 2m$, we cannot uniquely retrieve $\boldsymbol{u}(n)$ from $\boldsymbol{q}(n)$. This makes it nontrivial to reconstruct $\hat{x}(n)$ from $q_l(n)$. Such reconstruction is however rendered possible by the fact that the $2m$ signals $u_l(n)$ are not independent, because the decimators in Fig. 8(a) are only m fold devices. It is easily verified that $u_l(n)$ satisfy

$$U_{m+l}(z)/U_l(z) = z^{-1}G_{m+l}(z^2)/G_l(z^2), \qquad 0 \leqslant l \leqslant m-1. \tag{26}$$

Starting from the m transmitted signals $q_l(n)$, let us first define the $2m$-vector $\boldsymbol{p}(n)$

$$\begin{aligned} [p_0(n) \quad p_1(n) \cdots p_{2m-1}(n)]^T = \boldsymbol{p}(n) &= \boldsymbol{C}^T\boldsymbol{q}(n) \\ &= \boldsymbol{C}^T\boldsymbol{C}\boldsymbol{u}(n). \end{aligned} \tag{27}$$

As such we cannot solve for $\boldsymbol{u}(n)$ from (27) since $\boldsymbol{C}^T\boldsymbol{C}$ is singular (because it is $2m \times 2m$ with rank not exceeding m). We will however invoke (26) in the next section to reconstruct $\boldsymbol{u}(n)$ from $\boldsymbol{p}(n)$, i.e., from the m transmitted signals $q_l(n)$.

IV. SYNTHESIS FILTERS FOR THE ALIAS-FREE m-BAND DSB QMF BANK

The analysis filters $h_k^{\text{DSB}}(n)$ are derived from the GDFT filters as in (19), which in turn is derived from the DFT prototype filter $h_0^{\text{DFT}}(n)$ as in (16). We assume that $h_0^{\text{DFT}}(n)$ is a real and symmetric sequence of length N so that $h_0^{\text{DFT}}(n) = h_0^{\text{DFT}}(N-1-n)$. This symmetry ensures that $H_0^{\text{DFT}}(z)$ is a linear phase filter, i.e., $H_0^{\text{DFT}}(e^{j\omega}) = e^{-j\omega(N-1)/2}H_{0,r}^{\text{DFT}}(e^{j\omega})$, where $H_{0,r}^{\text{DFT}}(e^{j\omega})$ is real. Accordingly, the GDFT analysis filters have response

$$H_k^{\text{GDFT}}(e^{j\omega}) = W_r^{-(l_0+((N-1)/2))(k+k_0)}e^{-j\omega(N-1)/2} \cdot H_{0,r}^{\text{DFT}}(e^{j\omega}W_r^{k+k_0}) \tag{28}$$

which represents a linear-phase frequency response except for the phase offset caused by $W_r^{-(l_0+(N-1)/2)(k+k_0)}$. The phase offset can be canceled by choosing $l_0 = -(N-1)/2$.

If the uniform DFT alias free QMF bank is such that the distortion function $T^{\text{DFT}}(z)$ has zeros on the unit circle, the reconstructed signal $\hat{x}(n)$ suffers from severe amplitude distortion. Such 'singularity situations' [13] in the DFT QMF bank imply singularities in the GDFT QMF bank derived from it (by (15)), which in turn imply singularities of the derived DSB QMF bank. Singularities cannot be avoided unless the quantities $N-1$ and r are of opposite parity (i.e., $N-1$ is odd if r is even and vice versa), as shown in [13]. Since we have $r = 2m$, we shall therefore take $N-1$ to be odd. Even with this choice, our experience shows that the DSB bank has singularities unless θ_k is chosen carefully. The choice $\theta_k = \pi(m+k_0)/2$, which is suggested in [8] will be made in this paper, as this is *one* way to avoid singularities.

In summary, for the rest of the paper we shall assume

$$r=2m,\quad k_0=m+\frac{1}{2},\quad l_0=-\frac{N-1}{2},\quad N-1=\text{odd},$$
$$\theta_k=\frac{\pi}{2}(m+k_0). \tag{29}$$

With the choice (29), we have $W_r^{k_0 r}=-1$ hence

$$H_0^{\mathrm{DRT}}(z)=\sum_{l=0}^{r-1}G_l(-z^r)z^{-l}. \tag{30}$$

In general, $N-1$ is not a multiple of r, so we can always find unique integers m_0, m_1 such that $N-1=m_0+m_1 r$, $0\leqslant m_0 \leqslant r-1$. From (22) we can now obtain

$$[C^TC]_{kl}=\frac{1}{2m^2}\sum_{n=0}^{m-1}\left\{\cos\frac{2\pi}{2m}(n+k_0)(k+l+m-N+1)\right.$$
$$\left.+\cos\frac{2\pi}{2m}(n+k_0)(k-l)\right\}. \tag{31a}$$

By using the identity $\Sigma_{n=0}^{m-1}\cos(2\pi/2m)a(n+m+1/2)= mW_r^{a(m+(1/2))}\delta((a))$ where a is any integer,[1] this can be simplified to yield

$$[C^TC]_{kl}=\frac{1}{2m}\left\{W_r^{(k+l+m-N+1)k_0}\delta((k+l+m-N+1))\right.$$
$$\left.+W_r^{(k-l)k_0}\delta((k-l))\right\} \tag{31b}$$

with k_0 as in (29). Thus $2mC^TC=I_{2m}+K$ where K is a $2m\times 2m$ matrix whose form depends on the relative values of m_0 and m. To be more specific, we have

$$K=\begin{bmatrix}(-1)^{m_1-1}J_{m_0+m+1} & 0\\ 0 & (-1)^{m_1}J_{m-m_0-1}\end{bmatrix},\quad \text{for } m_0<m \tag{32}$$

and

$$K=\begin{bmatrix}(-1)^{m_1}J_{m_0-m+1} & 0\\ 0 & (-1)^{m_1-1}J_{3m-m_0-1}\end{bmatrix},\quad \text{for } m_0\geqslant m \tag{33}$$

where J_k denotes the $k\times k$ permutation matrix defined in Section I. The components of $p(n)$ and $u(n)$ are therefore related in the z-domain by

$$2mP_l(z)=U_l(z)+s(l)(-1)^{m_1}U_{((m_0-m-l))}(z),\quad 0\leqslant l\leqslant 2m-1 \tag{34}$$

where $((\cdot))$ is abbreviation for "mod $2m$." Here $s(l)$ is a sign parameter defined as

$$s(l)=\begin{cases}-1, & \text{for } 0\leqslant l\leqslant m_0+m\\ 1, & \text{for } m_0+m<l\leqslant 2m-1\end{cases} \tag{35}$$

for $m_0<m$ and

$$s(l)=\begin{cases}1, & \text{for } 0\leqslant l\leqslant m_0-m\\ -1, & \text{for } m_0-m<l\leqslant 2m-1\end{cases} \tag{36}$$

for $m_0\geqslant m$. Equation (34) can be written as two sets of equations:

$$2mP_l(z)=U_l(z)+s(l)(-1)^{m_1}U_{((m_0-m-l))}(z) \tag{37}$$
$$2mP_{l+m}(z)=U_{l+m}(z)+s(l+m)(-1)^{m_1}U_{((m_0-l))}(z) \tag{38}$$

with $0\leqslant l\leqslant m-1$. We know, $U_{m+l}(z)$ can be written in terms of $U_l(z)$ using (26). Similarly $U_{((m_0-l))}(z)$ can be written in terms of $U_{((m_0-m-l))}(z)$ because $((m_0-m-l))-((m_0-l))=\pm m$, but the relation depends on whether $((m_0-m-l))$ is greater or less than $((m_0-l))$. Accordingly, define a new sign parameter,

$$\epsilon(l)=\begin{cases}-1, & \text{for } ((m_0-l))>((m_0-m-l))\\ 1, & \text{for } ((m_0-l))<((m_0-m-l))\end{cases}. \tag{39}$$

We then have

$$U_{((m_0-l))}(z)=z^{\epsilon(l)}\frac{G_{((m_0-l))}(z^2)}{G_{((m_0-m-l))}(z^2)}U_{((m_0-m-l))}(z). \tag{40}$$

Substituting (26), (40) in (37), (38), we obtain

$$2mP_l(z)=U_l(z)+s(l)(-1)^{m_1}U_{((m_0-m-l))}(z) \tag{41}$$
$$2mP_{l+m}(z)=z^{-1}\frac{G_{m+l}(z^2)}{G_l(z^2)}U_l(z)+s(m+l)(-1)^{m_1}z^{\epsilon(l)}$$
$$\cdot\frac{G_{((m_0-l))}(z^2)}{G_{((m_0-m-l))}(z^2)}U_{((m_0-m-l))}(z),\quad \text{for } 0\leqslant l\leqslant m-1. \tag{42}$$

This can be solved for $U_l(z)$ to yield

$$U_l(z)=S_l(z)P_l(z)+S_{l+m}(z)P_{l+m}(z),\quad \text{for } 0\leqslant l\leqslant m-1 \tag{43}$$

with

$$S_l(z)=\frac{2ms(m+l)z^{\epsilon(l)}G_l(z^2)G_{((m_0-l))}(z^2)}{s(m+l)z^{\epsilon(l)}G_{((m_0-l))}(z^2)G_l(z^2)-z^{-1}s(l)G_{m+l}(z^2)G_{((m_0-m-l))}(z^2)} \tag{44}$$

$$S_{l+m}(z)=\frac{-2ms(l)G_l(z^2)G_{((m_0-m-l))}(z^2)}{s(m+l)z^{\epsilon(l)}G_{((m_0-l))}(z^2)G_l(z^2)-z^{-1}s(l)G_{m+l}(z^2)G_{((m_0-m-l))}(z^2)}. \tag{45}$$

[1] Here $\delta((a))$ is defined to be '1' if $a=0 \bmod 2m$ and '0' otherwise.

In summary, we have been able to express the $2m$ signals $U_l(z)$ in terms of the m signals $Q_l(z)$ transmitted by the m-band DSB QMF bank. The signal $\hat{X}(z)$ in Fig. 7 can be expressed in terms of $U_l(z)$ as

$$\hat{X}(z) = \sum_{l=0}^{2m-1} R_l(z^{2m}) z^{-(2m-1-l)} U_l(z^m) \tag{46}$$

which can be rewritten using (26) as

$$\hat{X}(z) = \sum_{l=0}^{m-1} \frac{R_l(z^{2m})G_l(z^{2m}) + R_{m+l}(z^{2m})G_{m+l}(z^{2m})}{G_l(z^{2m})} z^{-(2m-1-l)} U_l(z^m). \tag{47}$$

Since (23) should be satisfied to cancel aliasing, this simplifies to

$$\hat{X}(z) = 2\sum_{l=0}^{m-1} \frac{S(z^{2m})}{G_l(z^{2m})} z^{-(2m-1-l)} U_l(z^m). \tag{48}$$

Substituting from (43) this can be rewritten in terms of $P_l(z)$ as

$$\hat{X}(z) = \sum_{l=0}^{2m-1} A_l(z^{2m}) z^{-(2m-1-l)} P_l(z^m) \tag{49}$$

where

$$A_l(z^{2m}) = 2S(z^{2m})S_l(z^m)/G_l(z^{2m}),$$
$$A_{m+l}(z^{2m}) = 2z^{-m}S(z^{2m})S_{m+l}(z^m)/G_l(z^{2m}),$$
$$\text{for } 0 \leq l \leq m-1. \tag{50}$$

Fig. 8(b) shows the synthesis bank for the m-channel DSB QMF bank, which should be used in conjunction with the analysis bank of Fig. 8(a). The synthesis filter $F_k^{\mathrm{DSB}}(z)$ which operates on the transmitted signal $q_k(n)$ is given by

$$F_k^{\mathrm{DSB}}(z) = \frac{1}{m}\sum_{l=0}^{2m-1} A_l(z^{2m}) z^{-(2m-1-l)} \cdot \cos\frac{2\pi}{2m}\left(l - \frac{N-1}{2} + \frac{m}{2}\right)\left(k + m + \frac{1}{2}\right). \tag{51}$$

The impulse response corresponding to (51) is given by (see Appendix C, Property 6)

$$f_k^{\mathrm{DSB}}(n) = \frac{1}{m} f(n) \cos\frac{2\pi}{2m}\left(n + \frac{N-1}{2} - \frac{m}{2} + 1 - 2m\right) \cdot \left(k + m + \frac{1}{2}\right) \tag{52}$$

where $f(n)$ is the impulse response of a "prototype" filter with transfer function

$$F(z) = \sum_{l=0}^{2m-1} A_l(-z^{2m}) z^{-(2m-1-l)}. \tag{53}$$

We have thus shown that the synthesis filters $f_k^{\mathrm{DSB}}(n)$ are cosine modulated versions of a real-valued response $f(n)$. With the analysis and synthesis filters $H_k^{\mathrm{DSB}}(z)$ and $F_k^{\mathrm{DSB}}(z)$ chosen in this manner, the output $\hat{X}(z)$ in Fig. 8(b) is same as $\hat{X}(z)$ in Fig. 7 except for a possible scale factor.

V. Simplification of the Formulas for the Synthesis Filters

The synthesis transfer functions (51) can be implemented in polyphase form by directly implementing the structure of Fig. 8(b). The expressions for $A_l(z)$ are given by (50), where $S_l(z)$ come from (44), (45). These expressions are complicated because of the presence of m_0, which was defined to be $(N-1) \bmod 2m$. It however turns out that these expressions can be simplified into a form which is completely free from m_0! It can be shown after considerable algebra, that $A_l(z)$ takes on the much simpler form

$$A_l(z) = \frac{4mS(z)\tilde{G}_l(z)}{\tilde{G}_l(z)G_l(z) + \tilde{G}_{((m+l))}(z)G_{((m+l))}(z)}, \quad 0 \leq l \leq 2m-1. \tag{54}$$

(Recall that the notation $\tilde{F}(z)$ stands for $F(z^{-1})$ in this context.) A detailed derivation of the simplified form (54) can be found in [16]. In what follows, we include a brief outline of the derivation, which can be skipped on first reading. Given the m-band DSB QMF bank, let us define an "equivalent primed system" by introducing delays into the analysis and synthesis banks. More specifically, let the new analysis and synthesis filters be

$$H_k'^{\mathrm{DSB}}(z) = z^{-\beta} H_k^{\mathrm{DSB}}(z),$$
$$F_k'^{\mathrm{DSB}}(z) = z^{-(2m-\beta)} F_k^{\mathrm{DSB}}(z). \tag{55}$$

If the outputs of the original and primed QMF banks are respectively $\hat{X}(z)$ and $\hat{X}'(z)$, then $\hat{X}'(z) = z^{-2m}\hat{X}(z)$, hence the alias-free property continues to hold, and the distortion functions are related to $T'(z) = z^{-2m}T(z)$. The impulse responses of the analysis filters of the original and primed systems are, respectively,

$$h_k^{\mathrm{DSB}}(n) = 2h_0^{\mathrm{DFT}}(n)\cos\frac{2\pi}{2m}\left(n - \frac{N-1}{2} + \frac{m}{2}\right) \cdot \left(k + m + \frac{1}{2}\right) \tag{56}$$

$$h_k'^{\mathrm{DSB}}(n) = 2h_0'^{\mathrm{DFT}}(n)\cos\frac{2\pi}{2m}\left(n - \frac{N'-1}{2} + \frac{m}{2}\right) \cdot \left(k + m + \frac{1}{2}\right) \tag{57}$$

where $h'^{\mathrm{DFT}}(n) = h^{\mathrm{DFT}}(n-\beta)$. Since $H_k^{\mathrm{DSB}}(z)$ have been obtained by combining the GDFT filters $H_k^{\mathrm{GDFT}}(z)$

and $H^{\mathrm{GDFT}}_{2m-1-k}(z)$ as in (19), we define the underlying GDFT filters for the primed system to be $H_k'^{\mathrm{GDFT}}(z) = z^{-\beta}H_k^{\mathrm{GDFT}}(z)$.

The synthesis bank of the primed system is also as in Fig. 8(b) with $A_l(z)$ replaced with $A_l'(z)$ and with C appropriately replaced. We would like to choose β such that, for the primed system, the quantity m_0 is equal to $2m-1$, which enables us to simplify the expressions for $A_l'(z)$. After this, it only remains to relate the expressions for $A_l(z)$ and $A_l'(z)$.

For the unprimed system, the GDFT filters $H_k^{\mathrm{GDFT}}(z)$ are given by (8) (with $r = 2m$ and with the superscripts on $G_l(z)$ deleted for simplicity). Similarly, for the primed system[2]

$$H_k'^{\mathrm{GDFT}}(z) = W_r^{-l_0'(k+k_0)} \sum_{l=0}^{2m-1} G_l'(z^{2m}) z^{-l} W_r^{-l(k+k_0)}. \tag{58}$$

Because of the above relations, we obtain

$$G_l'(z) = \begin{cases} -z^{-1}G_{l+2m-\beta}(z), & 0 \leqslant l \leqslant \beta-1 \\ G_{l-\beta}(z), & \beta-1 < l \leqslant 2m-1 \end{cases} \tag{59}$$

by equating like powers of z. It is immediately clear that the quantities $R_l'(z)$ for the alias free primed GDFT system are given by

$$R_l'(z) = \begin{cases} -R_{l+2m-\beta}(z), & 0 \leqslant l \leqslant \beta-1 \\ z^{-1}R_{l-\beta}(z), & \beta-1 < l \leqslant 2m-1 \end{cases} \tag{60}$$

so that $S'(z) = G_l'(z)R_l'(z) = z^{-1}S(z)$ is independent of l. Notice that this relation is compatible with the relation $T'(z) = z^{-2m}T(z)$.

For the primed alias-free DSB system, the expression for $A_l'(z)$ can be obtained by modifying (50), (44) and (45) appropriately. Since $m_0' = 2m-1$, we have from (35), (36) and (39), $s'(l) = -s'(l+m) = -\epsilon'(l) = 1$ for $0 \leqslant l \leqslant m-1$. This results in

$$A_l'(z) = 4mS'(z)G'_{2m-1-l}(z)/D_l'(z)$$
$$A'_{m+l}(z) = 4mS'(z)G'_{m-1-l}(z)/D_l'(z) \tag{61}$$

for $0 \leqslant l \leqslant m-1$ where $D_l'(z) = G_l'(z)G'_{2m-1-l}(z) + G'_{m+l}(z)G'_{m-1-l}(z)$. Since $h_0^{\mathrm{DFT}}(n)$ is a symmetric sequence, and since it is related to $G_l(z)$ as in (30), $G_l(z)$ and $G_l'(z)$ satisfy (A5) (Appendix C, Property 3). Thus with $m_0' = 2m-1$ we do have $G_l'(z) = (-1)^{m_1}z^{-m_1}G'_{2m-1-l}(z^{-1})$, for $0 \leqslant l \leqslant 2m-1$ which gives us

$$G'_{2m-1-l}(z) = (-1)^{m_1}z^{-m_1}G_l'(z^{-1}), G'_{m-1-l}(z)$$
$$= (-1)^{m_1}z^{-m_1}G'_{m+l}(z^{-1}) \tag{62}$$

[2] The quantities k_0, l_0, θ_k for the two systems are related by $k_0' = k_0$, $l_0' = l_0 - \beta$, and $\theta_k' = \theta_k$. We also have $m_0' = 2m-1$, $m_1' = m_1$.

whence

$$D_l'(z) = \tilde{G}_l'(z)G_l'(z) + \tilde{G}'_{m+l}(z)G'_{m+l}(z) \tag{63}$$
$$A_l'(z) = 4mS'(z)\tilde{G}_l'(z)/D_l'(z),$$
$$A'_{m+l}(z) = 4mS'(z)\tilde{G}'_{m+l}(z)/D_l'(z) \tag{64}$$

for $0 \leqslant l \leqslant m-1$. The two expressions in (64) can be combined into

$$A_l'(z) = \frac{4mS'(z)\tilde{G}_l'(z)}{\tilde{G}_l'(z)G_l'(z) + \tilde{G}'_{((m+l))}(z)G'_{((m+l))}(z)},$$
$$0 \leqslant l \leqslant 2m-1. \tag{65}$$

It only remains to relate $A_l'(z)$ to $A_l(z)$. For this notice that $F_k^{\mathrm{DSB}}(z)$ is given by (51) whereas $F_k'^{\mathrm{DSB}}(z)$ is given by a similar expression,

$$F_k'^{\mathrm{DSB}}(z) = \frac{1}{m}\sum_{l=0}^{2m-1} A_l'(z^{2m})z^{-(2m-1-l)} \cdot \cos\frac{2\pi}{2m}\left(l - \frac{N'-1}{2} + \frac{m}{2}\right)\left(k+m+\frac{1}{2}\right). \tag{66}$$

From the three relations (51), (55), and (66) we obtain [16]

$$A_l(z) = \begin{cases} zA'_{\beta+l}(z), & 0 \leqslant l \leqslant 2m-1-\beta \\ -A'_{\beta+l-2m}(z), & 2m-1-\beta < l \leqslant 2m-1. \end{cases} \tag{67}$$

By substituting from (65) into (67), we can thus express $A_l(z)$ in terms of $S'(z)$ and $G_l'(z)$. By further invoking the relation (59) we finally arrive at [16]

$$A_l(z) = 4mS(z)\tilde{G}_l(z)/D_l(z),$$
$$A_{m+l}(z) = 4mS(z)\tilde{G}_{m+l}(z)/D_l(z), \tag{68}$$
$$D_l(z) \triangleq \tilde{G}_l(z)G_l(z) + \tilde{G}_{m+l}(z)G_{m+l}(z). \tag{69}$$

for $0 \leqslant l \leqslant m-1$. The final expression (54) follows from this.

VI. FIR and IIR Synthesis Filters for the m-Band DSB QMF Bank

Summarizing our results so far, Fig. 8(a) is the analysis bank for the m-band DSB QMF system, with the analysis filters given by (20), (21). The impulse responses (20) are cosine modulated versions of a real-coefficient prototype $h_0^{\mathrm{DFT}}(n)$. The synthesis bank which gives rise to alias-free reconstruction is as in Fig. 8(b) where $A_l(z)$ are given by (54). In this expression, $G_l(z)$ stand for the same $G_l^{\mathrm{GDFT}}(z)$ in Fig. 8(a), with superscripts deleted for simplicity. The synthesis filters $F_k^{\mathrm{DSB}}(z)$ can be written in terms of $A_l(z)$ as in (51), and the impulse responses $f_k^{\mathrm{DSB}}(n)$ are cosine-modulated versions of the synthesis prototype $f(n)$ as in (52), with the prototype transfer function $F(z)$ as given in (53). All the impulse response coefficients $h_k^{\mathrm{DSB}}(n)$ and $f_k^{\mathrm{DSB}}(n)$ are real-valued since the prototypes $h_0^{\mathrm{DFT}}(n)$ and $f(n)$ are real. The alias-free system has distortion transfer function $T^{\mathrm{DSB}}(z)$ given by (24) where $S(z)$ is the l-invariant product (23). This same $S(z)$ appears also in the expression (54).

Recall that, we restricted $H_0^{\mathrm{DFT}}(z)$ to be FIR, hence $H_k^{\mathrm{DSB}}(z)$ are FIR. However, the synthesis filters $F_k^{\mathrm{DSB}}(z)$ can be FIR or IIR, depending on the choice of $S(z)$ in (54). For a given analysis bank, we have considerable freedom over the choice of $S(z)$ (because, given the set of functions $G_l(z)$, we can pick all $R_l(z)$ according to (23) for many choices of $S(z)$). This freedom can be exploited in order to obtain either FIR or stable IIR synthesis filters, as shown next. In any event, since $T^{\mathrm{DSB}}(z)$ should not have zeros on the unit circle, $S(z)$ should in turn be free from such zeros.

Since $\tilde{G}_l(z)$ is the complex conjugate of $G_l(z)$ on the unit circle, $D_l(z)$ is a real nonnegative function for $z=e^{j\omega}$. Moreover, since $S(z)$ should be free from zeros on the unit circle, $G_l(z)$ should be free of such zeros (i.e., the analysis bank must be designed with this constraint in mind; it is for this reason that we picked $N-1$ to be odd; see also [13]). In other words, $D_l(e^{j\omega})$ is strictly positive for all ω. Moreover, the zeros of $D_l(z)$ are in reciprocal pairs because of the form (69).

FIR Synthesis Filters: If we wish to eliminate phase distortion completely, $T^{\mathrm{DSB}}(z)$ should have linear-phase, and this can be accomplished when $F_k^{\mathrm{DSB}}(z)$ are chosen as appropriate FIR functions. Suppose we take $S(z)$ to be

$$S(z)=\prod_{l=0}^{m-1} z^{-p(l)}D_l(z) \tag{70}$$

(where $p(l)$ is a large enough integer such that there are no positive powers of z in (70)), then $A_l(z)$ are polynomials in z^{-1} hence $F_k^{\mathrm{DSB}}(z)$ are causal FIR filters. The overall distortion transfer function is given by $T^{\mathrm{DSB}}(z)=2z^{-(2m-1)}\prod_{l=0}^{m-1}z^{-2mp(l)}D_l(z^{2m})$. This is FIR and moreover, does not have zeros on the unit circle. Since $D_l(z)$ has zero occurring in reciprocal pairs, $S(z)$ given by (70) (and hence $T^{\mathrm{DSB}}(z)$) has linear phase, which means phase distortion has been eliminated (residual amplitude distortion can be equalized, as demonstrated in the next section).

Further Properties of FIR Synthesis Banks: Since the prototype $H_0^{\mathrm{DFT}}(z)$ satisfies $h_0^{\mathrm{DFT}}(n)=h_0^{\mathrm{DFT}}(N-1-n)$, the functions $G_l(z)$ satisfy (A5) (Appendix C, Property 3). So, the analysis bank of Fig. 8(a) can be implemented more efficiently by sharing the multiplier coefficients of the pairs $\{G_l(z), G_{((m_0-l))}(z)\}$. It can be shown [16] that the functions $A_l(z)$ in the corresponding synthesis bank of Fig. 8(b) also satisfy a similar relation, namely

$$A_l(z)=\begin{cases}(-1)^{l_1}z^{-l_1}\tilde{A}_{m_0-l}(z), & l\leqslant m_0\\ (-1)^{l_1+1}z^{-(l_1+1)}\tilde{A}_{2m+m_0-l}(z), & l>m_0\end{cases} \tag{71}$$

for $0\leqslant l\leqslant 2m-1$, where $l_1=2\sum_{l=0}^{m-1}p(l)-m_1$. Accordingly, a factor-of-two savings in the number of multipliers can be obtained while implementing Fig. 8(b). One further consequence [16] of the symmetry relation (71) is that, the synthesis prototype $F(z)$ has linear phase. More specifically, the corresponding impulse response coefficients satisfy

$$f(n)=f(L-1-n)$$

where

$$L-1=2Jm-(N-1)+2(2m-1),\ J=2\sum_{l=0}^{m-1}p(l).$$

Comments on Filter Length: It should be noticed that the FIR functions $A_l(z)$ defined in (54) with $S(z)$ as in (70), have higher order than $G_l(z)$. Accordingly, the FIR synthesis filters are longer than the analysis filters, which is a price paid for the exact alias cancellation property. However, many of the trailing coefficients in the impulse response $f(n)$ are extremely small numbers and can be discarded. A practical question here is, how many of the coefficients in the synthesis prototype $f(n)$ should be retained so as to keep aliasing at a reasonable level (for example, the amount permitted in [8], [9])? Such issues, and a performance-comparison with the results in [8], [9] are currently under study.

IIR Synthesis Filters: If we wish to eliminate amplitude distortion (rather than phase distortion) completely, $T^{\mathrm{DSB}}(z)$ must be forced to be IIR all pass. For this, the choice of $F_k^{\mathrm{DSB}}(z)$ and hence $A_l(z)$ must be IIR too. Now, since $D_l(e^{j\omega})$ is strictly positive, we can write $D_l(z)=\tilde{E}_l(z)E_l(z)$ where $E_l(z)$ is the minimum-phase spectral factor (i.e., has all zeros strictly inside the unit circle). If we now choose $S(z)=\prod_{l=0}^{m-1}z^{-p(l)}\tilde{E}_l(z)/E_l(z)$ then we have $T^{\mathrm{DSB}}(z)=2z^{-(2m-1)}\prod_{l=0}^{m-1}z^{-2mp(l)}\tilde{E}_l(z^{2m})/E_l(z^{2m})$ which is stable all pass. Thus the DSB QMF system is free from amplitude distortion. The expressions for $A_l(z)$ are now given by

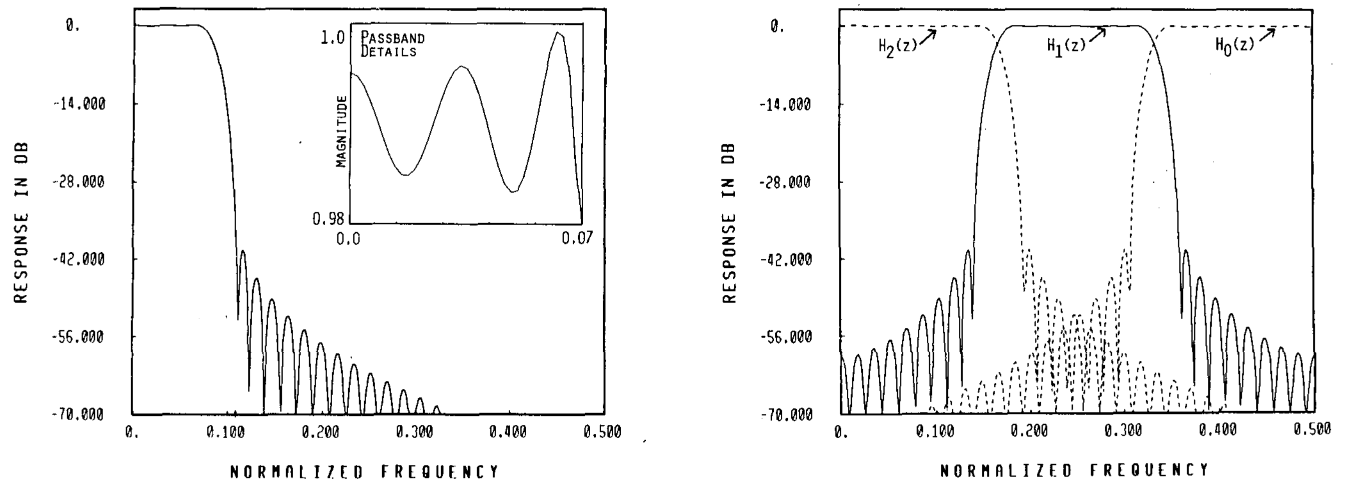
$$A_l(z)=\frac{4mz^{-p(l)}\tilde{G}_l(z)}{E_l^2(z)}\prod_{\substack{k=0\\k\neq l}}^{m-1}\frac{z^{-p(k)}\tilde{E}_k(z)}{E_k(z)},$$

$$A_{m+l}(z)=\frac{4mz^{-p(l)}\tilde{G}_{m+l}(z)}{E_l^2(z)}\prod_{\substack{k=0\\k\neq l}}^{m-1}\frac{z^{-p(k)}\tilde{E}_k(z)}{E_k(z)} \tag{72}$$

which are clearly stable IIR functions. The synthesis filters (51) are, therefore, stable, and are completely defined.

VII. A Design Example

Consider a three-channel DSB QMF bank ($m=3$). The analysis filters $h_k^{\mathrm{DSB}}(n)$, $0\leqslant k\leqslant 2$ are obtained as in (20), where $h_0^{\mathrm{DFT}}(n)$ is itself a linear-phase filter obtained using a standard Kaiser-window approach [3, 12], and has the following features: order $N-1=55$, stopband attenuation, $A_s=40$ dB, center of transition band $=0.178\pi$ and passband edge $=0.14\pi$. Notice that $\pi/2m\simeq 0.167\pi$ and is between the passband and stopband edges. Table I shows the coefficients $g_l(n)$ of the decimated transfer functions $G_l(z)$, (i.e., $G_l^{\mathrm{GDFT}}(z)$) which are related to $H_0^{\mathrm{DFT}}(z)$ by $H_0^{\mathrm{DFT}}(z)=\sum_{l=0}^{5}G_l(-z^6)z^{-l}$. Fig. 9(a) shows the prototype response, whereas Fig. 9(b) shows the analysis filter responses. The information in Table I enables us to obtain the coefficients of $D_l(z)$, $0\leqslant l\leqslant 2$ defined in (69), from which the coefficients of the polynomials $A_l(z)$ are calcu-



(a) (b)

Fig. 9. (a) Magnitude response of the analysis prototype $H_0^{\mathrm{DFT}}(z)$. (b) Magnitude responses for the analysis bank filters.

TABLE I
COEFFICIENTS OF DECIMATED FILTERS $G_l(z)$, i.e., $G_l^{\mathrm{GDFT}}(z)$

n	$g_0(n)$	$g_1(n)$	$g_2(n)$	$g_3(n)$	$g_4(n)$	$g_5(n)$
0	0.6068d-03	0.1813d-02	0.2833d-02	0.3115d-02	0.2224d-02	0.7567d-04
1	0.2910d-02	0.5827d-02	0.7500d-02	0.6881d-02	0.3495d-02	-0.2217d-02
2	0.8812d-02	0.1411d-01	0.1580d-01	0.1223d-01	0.3154d-02	-0.9798d-02
3	0.2320d-01	0.3247d-01	0.3296d-01	0.2127d-01	-0.3578d-02	-0.3953d-01
4	0.8161d-01	0.1228d+00	0.1557d+00	0.1739d+00	0.1739d+00	0.1557d+00
5	-0.1228d+00	-0.8161d-01	-0.3953d-01	-0.3578d-02	0.2127d-01	0.3296d-01
6	-0.3247d-01	-0.2320d-01	-0.9798d-02	0.3154d-02	0.1223d-01	0.1580d-01
7	-0.1411d-01	-0.8812d-02	-0.2217d-02	0.3495d-02	0.6881d-02	0.7500d-02
8	-0.5827d-02	-0.2910d-02	0.7567d-04	0.2224d-02	0.3115d-02	0.2833d-02
9	-0.1813d-02	-0.6068d-03	0. d+00	0. d+00	0. d+00	0. d+00

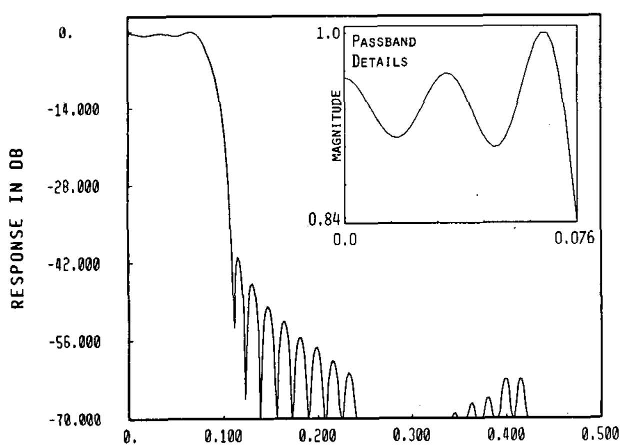


Fig. 10. Magnitude response of the synthesis prototype $F(z)$.

lated. The synthesis bank can now be implemented as in Fig. 8(b). The FIR synthesis prototype $F(z)$ has order 267. Fig. 10 show the magnitude response of $F(z)$. Many of the coefficients of the response $f(n)$ are very small. If we retain only the N dominant coefficients in $f(n)$, the result resembles those reported in [8], [9]. Further study is required to obtain a complete theoretical and experimental comparison of the methods reported here and in [8], [9].

We can find the coefficients of $F_k^{\mathrm{DSB}}(z)$ from (52). Finally the distortion function $T^{\mathrm{DSB}}(z)$ of the alias-free system can be computed as $T^{\mathrm{DSB}}(z)=\sum_{k=0}^{2}H_k^{\mathrm{DSB}}(z)F_k^{\mathrm{DSB}}(z)/3$ and has linear phase. The magnitude response $|T^{\mathrm{DSB}}(e^{j\omega})|$ is periodic with period $2\pi/6$, because $T^{\mathrm{DSB}}(z)=z^{-5}S(z^6)$. The broken curve in Fig. 11(a) shows a plot of $|S(e^{j\omega})|$. The amplitude distortion $|T^{\mathrm{DSB}}(e^{j\omega})|$ can be equalized by use of an FIR linear-phase equalizer $E(z^6)$ connected in cascade with the QMF bank. The function $E(z)$ can be designed by invoking McClellan–Parks' algorithm [11] with appropriate choice [12] of the "desired function" $D(e^{j\omega})$ and "weighting function" $W(e^{j\omega})$ as $D(e^{j\omega})=1/|S(e^{j\omega})|$ and $W(e^{j\omega})=|S(e^{j\omega})|$ for $0\leqslant\omega\leqslant\pi$, where $S(z)$ is defined in (70). We used a linear-phase function $E(z)$ of order 16. The solid curve in Fig. 11(a) shows the plot $|E(e^{j\omega})|$ whereas Table II shows the equalizer coefficients. The equalizer requires a total of nine multipliers. Fig. 11(b) shows the equalized response in an expanded scale. The equalized system has peak amplitude distortion < 0.014 dB, and is of course free from aliasing and phase distortion.

VIII. CONCLUDING REMARKS

The main purpose of this work has been to address the DSB QMF bank from a theoretical viewpoint, and to derive closed form expressions for the synthesis filters so

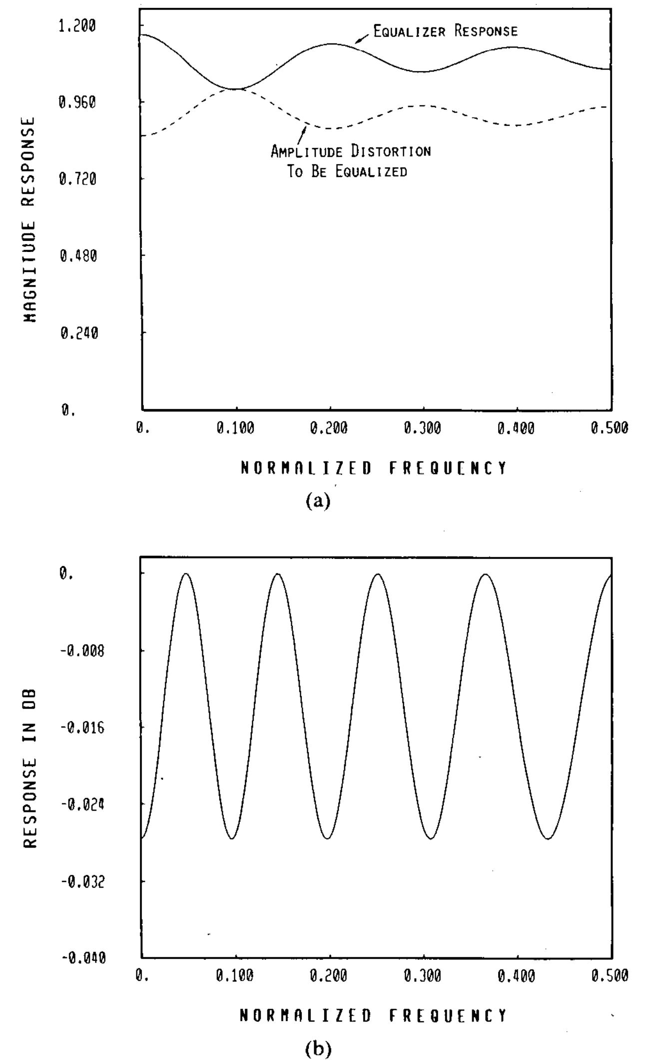


Fig. 11. (a) The distortion measure $|S(e^{j\omega})|$ and the equalizer response $|E(e^{j\omega})|$. (b) The equalized response $|E(e^{j\omega})S(e^{j\omega})|$.

TABLE II
THE DETAILS OF THE EQUALIZER, DESIGNED USING MCCLELLAN–PARKS PROGRAM

```
              finite impulse response (fir)
           linear phase digital filter design
                remez exchange algorithm

                     bandpass filter

                   filter length =  17

               ***** impulse response *****
           h( 1) =  0.71634218e-03 = h( 17)
           h( 2) =  0.57576527e-03 = h( 16)
           h( 3) =  0.53369938e-02 = h( 15)
           h( 4) =  0.27471144e-01 = h( 14)
           h( 5) =  0.89133037e-02 = h( 13)
           h( 6) =  0.43223598e-02 = h( 12)
           h( 7) = -0.91254851e-03 = h( 11)
           h( 8) = -0.48980727e-02 = h( 10)
           h( 9) =  0.10875202e+01 = h(  9)

                          band  1
lower band edge         0.
upper band edge         0.5000000
desired value           1.0000000
weighting               1.0000000
deviation               0.0015850
deviation in db         0.0137563

extremal frequencies--maxima of the error curve
   0.          0.0486111    0.0972223    0.1458334    0.1979167
   0.2500001   0.3055554    0.3645830    0.4305550    0.5000000
```

as to cancel aliasing completely. We have also indicated how phase distortion or amplitude distortion can be completely eliminated. The analysis and synthesis filters are cosine modulated versions of real-coefficient prototypes. Both the analysis and synthesis filters can be implemented in polyphase form, and moreover, if the analysis prototype has linear phase, there is an additional factor of 2 saving in the complexity of the analysis and synthesis banks. Further study is required, however, in order to obtain a complete comparison of the method with several other techniques reported in the recent past, for m-band QMF design [7]–[10], [17].

APPENDIX A

If $S_k(z) = S(z)$ for all k, then (3) clearly reduces to (4). Conversely, in order to cancel aliasing for all possible $x(n)$, we must have $\sum_{k=0}^{r-1} S_k(z^r) W_r^{-kl} = 0$ for $0 < l \leqslant r-1$. In other words, the r-point IDFT of $S_k(z^r)$ should be an impulse, which implies that $S_k(z^r)$ must be independent of k.

APPENDIX B

Consider *any* r-band maximally decimated QMF bank with analysis and synthesis filters $H_k(z), F_k(z)$, $0 \leqslant k \leqslant r-1$, where $r = 2m$. Assume that this bank is free from aliasing, so that we have the relations $\sum_{k=0}^{2m-1} H_k(zW_{2m}^{-l}) F_k(z) = 0$ for $1 \leqslant l \leqslant 2m-1$. If we now replace the $2m$-fold decimators and interpolators with m-fold devices, then the conditions for alias-cancelation are $\sum_{k=0}^{2m-1} H_k(zW_{2m}^{-2l}) F_k(z)$ with $1 \leqslant l \leqslant m-1$ which is a subset of the set of equations for the maximally decimated case. In other words, the alias free property remains intact. Moreover, the overall transfer function becomes $\sum_{k=0}^{2m-1} H_k(z) F_k(z)/m$ which is two times the distortion function of the maximally decimated bank.

APPENDIX C

Certain useful properties of analysis banks are summarized here. Detailed proofs of these can be found in [16].

Property 1: This is useful in studying analysis banks. Let $H(z)$ be any causal transfer function written in the polyphase form [3] $H(z) = \sum_{l=0}^{r-1} G_l(z^r) z^{-l}$. Then the impulse response $h(n)$ is symmetric, i.e., $h(n) = h(N-1-n)$ for some integer N if and only if

$$G_l(z) = \begin{cases} z^{-m_1}\tilde{G}_{m_0-l}(z), & 0 \leqslant l \leqslant m_0 \\ z^{-(m_1-1)}\tilde{G}_{r+m_0-l}(z), & m_0 < l \leqslant r-1 \end{cases} \tag{A1}$$

where m_0 and m_1 are integers such that $0 \leqslant m_0 \leqslant r-1$ and $N-1 = m_0 + m_1 r$. Under this condition, $H(z)$ is automatically FIR with order $N-1$ (or less). Moreover, the transfer functions $G_l(z)$ have order determined by

$$\text{Order of } G_l(z) \begin{cases} \leqslant m_1, & 0 \leqslant l \leqslant m_0 \\ \leqslant m_1 - 1, & m_0 < l \leqslant r-1. \end{cases} \tag{A2}$$

Property 2: The counterpart of the above result, useful for synthesis banks is this: let $F(z)$ be any causal transfer function written in the form $F(z) = \sum_{l=0}^{r-1} R_l(z^r) z^{-(r-1-l)}/r$. Then the impulse response satisfies symmetry, i.e., $f(n) = f(L-1-n)$ for some L if and only if

$$R_l(z) = \begin{cases} z^{-l_1}\tilde{R}_{l_0-l}(z), & 0 \leqslant l \leqslant l_0 \\ z^{-(l_1+1)}\tilde{R}_{r+l_0-l}(z), & l_0 < l \leqslant r-1 \end{cases} \tag{A3}$$

where l_0, l_1 are integers such that $0 \leqslant l_0 \leqslant r-1$ and $L-1 = (l_1+1)r + r - 2 - l_0$. Under this condition, $F(z)$ is automatically FIR with order $L-1$ (or less), and the order of $R_l(z)$ satisfies

$$\text{Order of } R_l(z) \begin{cases} \leqslant l_1, & 0 \leqslant l \leqslant l_0 \\ \leqslant l_1 + 1, & l_0 < l \leqslant r-1. \end{cases} \tag{A4}$$

Property 3: When we derive a uniform GDFT bank from an uniform DFT bank, there is a modification of Property 1, which turns out to be useful. Suppose $H(z)$ is some causal transfer function of the form $H(z) = \sum_{l=0}^{r-1} G_l(-z^r) z^{-l}$. Then the impulse response $h(n)$ of $H(z)$ satisfies $h(n) = h(N-1-n)$ for some integer N, if and only if

$$G_l(z) = \begin{cases} (-1)^{m_1} z^{-m_1}\tilde{G}_{m_0-l}(z), & 0 \leqslant l \leqslant m_0 \\ (-1)^{m_1-1} z^{-(m_1-1)}\tilde{G}_{r+m_0-l}(z), & m_0 < l \leqslant r-1 \end{cases} \tag{A5}$$

where m_0, m_1 are integers such that $N-1 = m_0 + m_1 r$, $0 \leqslant m_0 \leqslant r-1$. The order of $G_l(z)$ satisfies (A2), and $H(z)$ is automatically FIR with order $\leqslant N-1$. A counterpart of this result, which is useful for the corresponding synthesis bank is shown in the following property.

Property 4: Let $F(z)$ be a causal function expressed as $F(z) = \sum_{l=0}^{r-1} R_l(-z^r) z^{-(r-1-l)}/r$. Then its impulse response $f(n)$ satisfies $f(n) = f(L-1-n)$ if and only if

$$R_l(z) = \begin{cases} (-1)^{l_1} z^{-l_1}\tilde{R}_{l_0-l}(z), & 0 \leqslant l \leqslant l_0 \\ (-1)^{l_1+1} z^{-(l_1+1)}\tilde{R}_{r+l_0-l}(z), & l_0 < l \leqslant r-1 \end{cases} \tag{A6}$$

where l_0, l_1 are integers such that $0 \leqslant l_0 \leqslant r-1$ and $L-1 = (l_1+1)r + r - 2 - l_0$. Under this condition $F(z)$ is FIR with order $\leqslant L-1$. The order of $R_l(z)$ is bounded as in (A4).

Property 5: This pertains to cosine modulated analysis banks. Let $H_k(z)$ be a transfer function of the form $H_k(z) = \sum_{l=0}^{r-1} G_l(z^r) z^{-l} \cos(2\pi/r)(l+l_1)(k+k_0)$ where k_0 is a half-integer (i.e., $k_0 - 0.5$ is integer), and l_1 is arbitrary. Then the impulse response $h_k(n)$ of $H_k(z)$ has the form $h_k(n) = h(n)\cos(2\pi/r)(l_1+n)(k+k_0)$ where $h(n)$ is the impulse response of the prototype $H(z)$ defined by $H(z) = \sum_{l=0}^{r-1} G_l(-z^r) z^{-l}$.

Property 6: A corresponding property which is useful in studying synthesis banks is this: Let $F_k(z) = \sum_{l=0}^{r-1} R_l(z^r) z^{-(r-1-l)} \cos(2\pi/r)(l+l_1)(k+k_0)/r$ where k_0 is a half integer. Then the impulse response $f_k(n)$ of $F_k(z)$ satisfies $f_k(n) = f(n)\cos(2\pi/r)(n-l_1-r+1)(k+k_0)$ where $f(n)$ is the impulse response of the prototype $F(z) = \sum_{l=0}^{r-1} R_l(-z^r) z^{-(r-1-l)}/r$.

Acknowledgment

The authors wish to thank Dr. R. E. Crochiere of the AT&T Bell Labs for encouraging discussions, and Dr. M. Vetterli of Columbia University, New York, for drawing our attention to [10] during our ICASSP '86 meeting.

References

[1] A. Croisier, D. Esteban, and C. Galand, "Perfect channel splitting by use of interpolation/decimation/tree decomposition techniques," presented at the Int. Symp. on Info., Circuits and Systems, Patras, Greece, 1976.

[2] D. Esteban and C. Galand, "Application of quadrature mirror filters to split-band voice coding schemes," in *Proc. IEEE Int. Conf. on Acoustics, Speech, and Signal Processing*, Hartford, CN, pp. 191–195, May 1977.

[3] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing.* Englewood Cliffs, NJ: Prentice Hall, 1983.

[4] C. R. Galand and H. J. Nussbaumer, "New quadrature mirror filter structures," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. ASSP-32, pp. 522–531, June 1984.

[5] T. H. Ramstad, "Analysis/synthesis filter banks with critical sampling," presented at the Int. Conf. on Digital Signal Processing, Florence, Sept. 1984.

[6] H. J. Nussbaumer and M. Vetterli, "Computationally efficient quadrature mirror filter banks," *Proc. IEEE Int. Conf. ASSP*, San Diego, CA, pp. 11.3.1–11.3.4, Mar. 1984.

[7] J. Masson and Z. Picel, "Flexible design of computationally efficient nearly perfect QMF banks," *Proc. IEEE Int. Conf. ASSP*, Tampa, FL, 1985.

[8] J. H. Rothweiler, "Polyphase quadrature filters, a new subband coding technique," *Proc. IEEE Int. Conf. on ASSP*, pp. 1980–1983, Boston, MA, Apr. 1983.

[9] P. L. Chu, "Quadrature mirror filter design for an arbitrary number of equal bandwidth channels," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, pp. 203–218, Feb. 1985.

[10] M. Vetterli, "Filter banks allowing for perfect reconstruction," in *Signal Processing*, vol. 10, no. 3, pp. 219–244, Apr. 1986.

[11] J. H. McClellan, T. W. Parks, and L. R. Rabiner, "FIR linear-phase filter design program," in *Programs for Digital Signal Processing*, IEEE Press, 1979.

[12] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*, Englewood Cliffs, NJ: Prentice Hall, 1983.

[13] K. Swaminathan and P. P. Vaidyanathan, "Theory and design of uniform DFT, parallel, quadrature mirror filter bank," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 1170–1191, Dec. 1986.

[14] M. J. T. Smith and T. P. Barnwell III, "A unifying framework for analysis/synthesis systems based on maximally decimated filter banks," in *Proc. IEEE Int. Conf. ASSP*, pp. 521–524, Tampa, FL, Mar. 1985.

[15] M. J. T. Smith and T. P. Barnwell III, "Exact reconstruction techniques for tree-structured subband coders," *IEEE Trans. Acoust., Speech, Signal Processing*, pp. 434–441, June 1986.

[16] P. P. Vaidyanathan and K Swaminathan, "Uniform DSB QMF banks that are entirely free from aliasing," *Tech. Rep.* CALTECH, Pasadena, CA, May 1986.

[17] P. P. Vaidyanathan, "Theory and design of M-channel maximally-decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property," *Tech. Rep.*, CALTECH, Pasadena, CA, Apr. 1986; also in *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 476–492, Apr. 1987.

[18] P. P. Vaidyanathan "A note on general parallel QMF banks," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 101–103, Jan. 1987.

P. P. Vaidyanathan (S'80–M'83), for a photograph and biography please see page 23 of the January 1987 issue of this TRANSACTIONS.

K. Swaminathan, photograph and biography not available at time of publication.